Exam 2: Solutions

1. State whether the statements are TRUE or FALSE. Justify your answer.

   (a) The set of correlated equilibria is the same as the set of Nash equilibria in the following game:

	L	R
T	$4,3$	$3,1$
M	$8,2$	$0,0$
B	$5,6$	$3,7$

   Solution. FALSE. There is a correlated equilibrium with support (ML, BL, BR). Let the corresponding probabilities for each action profile be (p_1, p_2, p_3), where the components sum up to 1. To see that it is a CE with a non-zero p_2, consider possible deviations. P1 does not want deviate from M and P2 does not want deviate from R because they get their maximal payoffs. There is no profitable deviation from B or L if

$$p_2 5 + p_3 3 \geq p_2 8 + p_3 0 \iff p_2 \leq p_3,$$
$$p_1 2 + p_2 6 \geq p_1 0 + p_2 7 \iff p_2 \leq 1/2 p_1.$$

   Hence, e.g. ML w.p. $1/2$, BL w.p. $1/4$, and BR w.p. $1/4$ is a CE but it is not a NE.

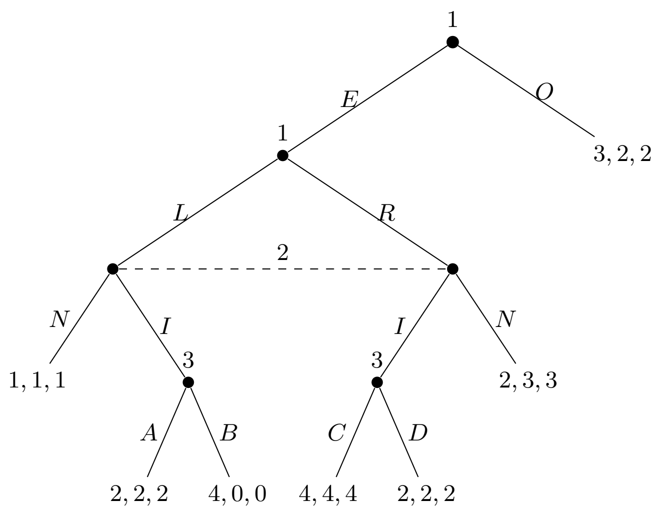
   (b) Consider infinitely repeated finite normal form stage game. If a feasible payoff vector v is such that each of its component is strictly larger than the smallest stage game Nash equilibrium payoff for that player, then there exists a $\delta \in (0,1)$ such that v is the payoff vector for some subgame perfect equilibrium when players discount with factor δ.

   Solution. TRUE. This is implied by the minmax folk theorem because a NE payoff is necessarily at least as large as the minmax payoff.

   (c) Asymmetric information may help to achieve a more efficient equilibrium (= better for all players) in some dynamic games.

   Solution. TRUE. E.g. reputation games.

   (d) There exists a belief μ such that $((OR, N, AC), \mu)$ is a sequential equilibrium in the following game:



Solution. FALSE. N is not sequentially rational with any beliefs when P3 plays AC.

2. This question considers the following game:

	L	R
U	$9,1$	$0,0$
D	θ,θ	$1,9$

(a) Find the unique Nash equilibrium of the game when $\theta = 10$.

Solution. NE: (D, L).

(b) Consider a situation where θ is 5 with probability 0.5 and 15 with probability 0.5. P2 does not know the true value of θ but P1 does. Find the unique pure strategy Bayes Nash equilibrium of the game.

Solution. BNE: (UD, L)

(c) Next, consider a situation where the information structure is otherwise as in (b) but P2 receives a signal $s \in \{h, l\}$ of θ: $Pr(s = h|\theta = 5) = 0$ and $Pr(s = h|\theta = 15) = 0.4$. Write the game as a game of incomplete information. Remember to include types for each player.

Solution.

- Players: P1 and P2
- Types: type for P1 is $\theta \in \{5, 15\}$ and type for P1 is $s \in \{h, l\}$
- Type distribution: $Pr(\theta = 5) = 0.5$ and $Pr(s = h|\theta = 5) = 0$ and $Pr(s = h|\theta = 15) = 0.4$ (equivalently $Pr(\theta = 5, s = h) = 0$, $Pr(\theta = 15, s = h) = 0.2$, $Pr(\theta = 5, s = l) = 0.5$, $Pr(\theta = 15, s = l) = 0.3$)
- Actions and payoffs are as in the question:

	L	R
U	$9,1$	$0,0$
D	θ,θ	$1,9$

(d) Show that the following two strategy profiles are Bayes Nash equilibria of the game in part (c): i) P1: D always, P2: R if $s = l$ and L if $s = h$; ii) P1: U if $\theta = 5$ and D if $\theta = 15$, P2: L always.

Solution. P2 updates by using Bayes rule: $Pr(\theta = 5|s = h) = 0$, $Pr(\theta = 5|s = l) = 5/8$.
Profile i): No profitable deviations by P1 when $\theta = 15$ because D is a dominant action. No profitable deviations when $\theta = 5$ either because $Pr(a_2 = L|\theta = 5) = Pr(s = h|\theta = 5) = 0$ and P1 gets a higher payoff from (DR) than from (UR).
No profitable deviations by P2 when $s = h$ because L is a dominant action. No profitable deviations when $s = l$ either if

$$9 \geq Pr(\theta = 5|s = l)5 + Pr(\theta = 15|s = l)15 = 5 * 5/8 + 15 * 3/8 = 70/8,$$

which is smaller than 9 and hence there are no profitable deviations and Profile i) is a BNE.
Profile ii): No profitable deviations by P1 when $\theta = 15$ because D is a dominant action. No profitable deviations when $\theta = 5$ either because P2 always plays L and P1 gets a higher payoff from (UL) than from (DL) when $\theta = 5$.
No profitable deviations by P2 when $s = h$ because L is a dominant action. No profitable deviations when $s = l$ either because

$$Pr(\theta = 5|s = l)1 + Pr(\theta = 15|s = l)15 \geq Pr(\theta = 5|s = l)0 + Pr(\theta = 15|s = l)9.$$

Hence, Profile ii) is a BNE.

3. Two firms are trying to force each other out of business and have started a price war. In each period $t = 0, 1, 2, ...$, the firms simultaneously choose between fighting (F) and quitting (Q). Quitting is irreversible and yields payoff 0 from that period onward. If both firms fight, they get payoff $-c$ per period. The game ends as soon as one of the firms quit and then the other firm receives a monopoly profit p per period from that period onward (including the period when the other firm quits). The players discount their costs and profits with discount factor δ. Assume that the firms are not liquidity constrained and can keep fighting forever.

(a) Consider a strategy profile $s_1(t) = F$ for all t and $s_2(t) = Q$ for all t. Is the profile a subgame perfect equilibrium?

Solution. This is a SPE: given the behavior of P1, P2 has no incentive to fight in one period (one-step deviation) because P1 will fight in every period. P1 gets utility $p/(1-\delta)$ so he has no incentive to deviate to get 0.

(b) Find a subgame perfect equilibrium where both players stop with the same constant probability in each period.

Solution. Let q be this probability of stopping. The condition for a mixed strategy equilibrium is that a player is indifferent between fighting and dropping out. In any period the expected payoff from fighting is $qp(1-\delta)^{-1} + (1-q)(-c)$. The continuation value is zero because the player will be indifferent in the next period: stopping gives a zero payoff, and hence the expected payoff after any action in the support of the mixed strategy is also zero. The utility from stopping is 0. Thus the equilibrium condition is

$$q\frac{p}{1-\delta} + (1-q)(-c) = 0 \iff q = \frac{c}{c + \frac{p}{1-\delta}}.$$

Hence, both players are indifferent in every period if the other player stops with probability q above. We can conclude that there is no profitable one-step deviations and hence both players stopping with the constant probability q is a SPE.

(c) Now, consider the same game but when Firm 2 observes the action taken by Firm 1 before taking its own action. Construct a subgame perfect equilibrium where Firm 1 randomizes in every period and Firm 2 randomizes if Firm 1 fights. Which firm gets a larger expected payoff in that equilibrium?

Solution. Let the stopping probability for Firm 1 be q_1 in every period. Firm 2 fights if Firm 1 stops. Let the stopping probability for Firm 2 be q_2 in every period when Firm 1 fights. We have the following indifference conditions:

$$\begin{aligned}
V_1 &= q_2\frac{p}{1-\delta} + (1-q_2)(-c) + \delta V_1 = 0 \iff q_2 = \frac{c}{c + \frac{p}{1-\delta}} \\
V_2 &= q_1\frac{p}{1-\delta} \\
&- c + \delta V_2 = 0 \iff q_1 = \frac{c}{\delta\frac{p}{1-\delta}}
\end{aligned}$$

Indifference guarantees that there are no profitable one-step deviations and hence the strategy profile is a SPE with expected payoffs $(0, c/\delta)$. Firm 2 is better off than Firm 1.

4. Consider the following (fictional) situation. A housing cooperative (*andelsboligforening*) has 9 identical apartments but only 3 parking slots that are allocated among the shareholders (=apartment owners) based on how long they have lived in the housing cooperative. Now, a new shareholder suggests that the housing cooperative increases the rent of the parking slots. The revenue from the rents is used to cover general expenses in the housing cooperative so that they benefit all shareholders equally. Does the shareholders' meeting approve the suggestion? The suggestion gets approved if a majority of the shareholders votes for it.

You are allowed to combine sub questions but then you need to state clearly which sub questions you are answering together.

(a) Define a game that describes the situation.

Solution (example).

- Players: P1, P2,..., P9 (9 shareholders)
- Actions: players simultanously choose I (vote for increase), N (vote for no increase)

- Payoffs: We allow payoffs to depend on how long each shareholder has lived in the cooperative as well as the voting decision. The rental price will be $\overline{p}$ if the number of I is at least 5 and it is $\underline{p} < \overline{p}$ the number of I is at most 4. Let $\alpha_i \in [0,1]$ be a parameter that captures the expected waiting time for each player until they would get a parking slot. We rearrange the players such that $\alpha_i \leq \alpha_j$ if $i > j$, i.e. smaller number players have lived in the cooperative the longest. Furthermore, let $\alpha_1 = \alpha_2 = \alpha_3 = 1$ to capture that there are 3 parking slots. Now the payoffs become:

$$(-\alpha_i + \frac{3}{9})p,$$

where $p = \overline{p}$ if the number of I is at least 5 and $p = \underline{p}$ if the number of I is at most 4.
Notice that we have left out the utility from the parking slot because it is unaffected by the price.

(b) Point out what assumptions you have made in part (a).

Solution (example). Assumptions:

- Players are otherwise homogeneous except that they differ in α and everyone wants to rent a parking slot if offered to do so.
- Players vote simultaneously (e.g. anonymous voting).
- Voting is about whether to increase the price by a given amount $\overline{p} - \underline{p}$, not about how large the increase should be.
- It is not possible to not to vote and everyone is present at the meeting.
- Voting decision does not affect other things, e.g. no conflict after split voting

(c) What would be a suitable solution concept to solve the game in part (a)? Argue why.

Solution (example). Can use NE because this is a simultaneous move game with complete information (players are heterogeneous but those differences are common knowledge).

(d) Write down the equations that characterize a solution (this means that a strategy profile that satisfies all of them is a solution).

(e) Either solve the game OR discuss what you would expect to happen in the game (the latter means writing a few sentences where you describe the main tradeoff).

Solution to (d) and (e) (example). Notice first that any voting profile where at least 6 players vote for the same alternative is a NE because then no one can change the outcome by voting differently. Players with $\alpha_i > 1/3$ want to have $\underline{p}$ and players with $\alpha_i < 1/3$ want to have $\overline{p}$. Therefore the NE in weakly dominant strategies is that all players with $\alpha_i > 1/3$ vote for N and all players with $\alpha_i < 1/3$ vote for I. Then the outcome is the high price if $\alpha_5 < 1/3$.

(f) Interpret your results (write a few sentences).

Solution. Because there are multiple equilibria, it is possible that majority shareholders fail to coordinate. However, if we restrict to the weakly dominant NE, the outcome represents the payoffs of the majority. If the current parking slot holders are likely to move away soon, α_4 and α_5 may be large enough that the majority votes for N in that equilibrium. However, if it is unlikely that at least two slots would become free any time soon, the majority prefers to increase the price.